Computational methods in engineering sciences, vol. 1, no. 3, Autumn 2023 Serial no. 3
DOI: 10.22034/cmes.2023.2018673.1019

The Spectral Characterization of n-Generalized Skew Projections

Seyedeh Sara Karimizad [1,*]

[1] Department of Basic Sciences, Ilam University, Ilam, Iran
E-mail addresses: s.karimizad@ilam.ac.ir
[*] Corresponding author

Received: 19/11/2023, Revised: 11/12/2023, Accepted: 19/12/2023.

Abstract
The discussion of n-generalized is a fundamental challenge in mathematical discussions. Considering that a lot of attention has been paid to this discussion in recent years and studies, it is clear that there are still many and various challenges in this field. In this paper we investigate and presented n-generalized skew projection for n=2,3 in Banach algebras. All the equations are given completely from the beginning and the basis, and the full explanation of each of the formulas along with their proof is presented in the paper. Also, all relevant issues have been fully discussed and discussed in the paper. Necessary and sufficient conditions for a linear combination of two n-generalized skew projection for n=2,3 to be a n-generalized skew projection for n= 2,3 are developed. Our results generalize results obtained for bounded linear operators on Hibert spaces. Also, all relevant issues have been fully discussed and discussed in the article.

Keywords
n-Generalized, skew projection, Mathematics

1. Introduction and preliminaries

A Throughout this paper, β will denote a complex unital Banach algebra with unit 1. If $\chi \in \beta$ then we denote the spectrum and the spectral radius of χ by $\sigma(\chi)$ and $\Gamma(\chi)$ respectively. $H(\beta)$ denotes the set of hermitian elements of β .An element is saidto be hermitian if $\|\exp(iah)\|$ =1 for all $\alpha \in$ □ .See [1, page 46]. For given $\alpha \in$ □ and $h \in H(B)$ symbols $\bar{\alpha}, h^*$ will mean the conjugate of α and the adjoint of h.It is well- known that a bounded linear operator T on a Hilbert space is hermitian if and only if T=T*.If $n \in$ □ and n>1, a n-generalized skew projection is an element $a \in M(\beta)$ such that an=-a*, where

(1.1)
$$M(\beta) = \{h + ik : \quad h, k \in H(\beta)\}$$

For properties and characterizations of generalized skew projection operators see [2,3].The purpose of the present paper is to consider the similar problem.

For n-generalized skew projections. Let $\beta =$ □ ^{3} with pointwise multiplication and let p: $\beta \to [0,\infty)$ be defined by

$$P(\alpha,\beta,\gamma) = \sup\{|\lambda^{-1}\alpha + \beta + \lambda\gamma| : \lambda \in \square, |\lambda| = 1\} \qquad \forall(\alpha,\beta,\gamma) \in \beta.$$

Define the norm $\|.\|$ on β by

$$\|\chi\| = \sup\{p(y\chi): y \in \beta, p(y) = 1\}$$

Then $(\beta, \|.\|)$ is a complex commutative Banach algebra with unit 1= (1,1,1,).

Since $H(\beta) \cap iH(\beta) = \{0\}$, hence each element of $M(\beta)$ has a unique representation of the form h+ik with h,k $\in H(\beta)$. Therefore we may define a linear involution * on H (β) by(h+ik)*=h-ik.

We say that $\alpha \in M(\beta)$ is normal if aa*= a*a. It is easy to see that a element a=h+ik $\in M(\beta)$ for some h,k $\in H(\beta)$ is normal if and only if hk=kh. We collect the basic propositions of $H(\beta)$ (for proofs see [2,pages 47 ,53,and54].

Proposition 1.1.

H (β) is closed real subspace of β and H (β) $\cap iH(\beta) = \{0\}$.

If h,k $\in H(\beta)$, then i(hk-kh) $\in H(\beta), \sigma(h) \subseteq R$ and $r(h) = \|h\|$.

Proposition 1.2.Suppose that $a = h + ik \in M(\beta)(h, k \in H(\beta)$is normal. Then;

$$\sigma(\chi) = \{\chi(x) : \chi \in \Delta_a\} \text{ for all } x \in \beta(a);$$

$\chi(h), \chi(k) \in R$ for all $\chi \in \Delta_a$;

$\chi(a^*) = \overline{\chi(a)}$ for all $\chi \in \Delta_a$, and

$\sigma(a^*) = \{\bar{\lambda} : \lambda \in \sigma(a)\}$;

if $x, y \in \beta(a)$, then $\sigma(x+y) \subseteq \sigma(x)+\sigma(y), \sigma(xy) \subseteq \sigma(x)\sigma(y)$.

Theorem1.3. Let that a=h+ik $\in M(B)$ is a3 – generalized skew projection Then;

(1) a is normal.

(2) $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \lambda^4 = -1\} \cup \{0\}$

(3) Moreover if hk $\in H(\beta)$, then

$$(a^*)^3 = -a \text{ and } a^9 = a.$$

Proof.

Since a3=-a* hence aa*= a(-a3)=(-a3)a=a*a.

n-GENERALIZED SKEW PROJECTIONS

Let $\lambda \in \sigma(a)$, then $\lambda = \chi(a)$ for some $\chi \in \Delta_a$, hence by proposition 1.3; $\bar{\lambda} = \chi(a^*) = \chi(-a^3) = -\chi(a)^3 = -\lambda^3$.If $\lambda \neq$, then $|\lambda| = 1$ and $\lambda^4 = \lambda\lambda^3 = \lambda(-\bar{\lambda} = -\lambda\bar{\lambda} = -1$.

From a3=-a*then we have; h3+h-3hk2=i(k3+k-3h2k)h3+h-3hk2=0 $\Rightarrow$ h=3hk2-h3,k3+k-3h2k=0 $\Rightarrow$ k=-k3+3h2k. Proposition 1.3 (4) and proposition 1.1.(2) show that σ (h3-3hk2+h),σ (k3+k-3h2k)$\subseteq$ R.Hence

$\sigma(k^3+k-3h^2k) \subseteq R \cap iR = \{0\}$.Thus $r(k^3+k-3h^2k) = 0$. Since $hk \in H(A)$, $k^3+k-3h^2k \in H(A)$.

Now use proposition 1.1 (2) to get k=3h2k-k3.Hence – h3+3hk2=h therefore (a*)3=(h-ik)3=h3-3h2ik+3h(ik)2-i3k3= h3-3hk2k-3hk2+ik3=h3-3hk2+i(-3h2k+k3)=-h-ik=-a and a9=(a3)3=(-a*)3=-(-a)=a.

Theorem 1.4 Suppose a=h+ik$\in$ M(β), and Suppose that

$$\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \lambda^4 = -1\} \cup \{0\}$$

Then we have

(i) If a is normal, then r (a3+a*)=0.

(ii) If hk, h2,k2 $\in$ H (β), Then a is 3-generalized skew projection.

Proof.Let b= a3+a*(1)Since a is normal, hk=kh,thus b $\in$ B(a).Take $\lambda \in \sigma(b)$. Then λ = $\chi(b) = \chi(a^2) + \chi(a^*)$ for some $\chi \in \Delta_a$.

Casel: $\chi(a) = 0$. Then we have $\lambda = 0$.

Case 2: $\chi(a) \neq 0$. Since $\chi(a) \in \sigma(a), \chi(a)^4 = -1$ it follows that $\lambda\chi(a) = \chi(a)^4 + \chi(\bar{a})\chi(a) = -1+1 = 0$ and so λ =0.Hence $\sigma(b) = \{0\}$. (2) Since h, k,h2,hk and k2 are all hermitian, it follow from [1,theorem 2.14] that hk=kh. Thus a is normal . From b= a3+a*=h3+3k2hi+3kh2i+k3-(h-ik) we get i (3k2h+3kh2-k)=b-(h3-k3-h).By proposition 1.1 and proposition 1.3

$\sigma(3k^2h+3hk^2-k) \subseteq R \cap iR = \{0\}$.Since $3k^2h+3hk^2-k \in H(A)$, $3k^2h+3hk^2-k = 0$, $b = h^3-k^3-h \in H(A)$. Since by (1),r(b)=0,we conclude that b=0, hence a3=-a*.

Theorem 1.5. Let a,b $\in M(B)(a+b \neq 0)$ be 3-generalized skew. Then a+b is a 3- generalized skew projection if and only if ab= ba =0.

Proof. We have (a+b)3=a3+3a2b+3ab2+b3=-a*+3a2b+3ab2-b*.If ab=ba=0 then a2b=a(ab)=0,ab2=(ab)b=0then (a+b)3=-a*-b*=-(a*+b*)=-(a+b)*hence a+b is a 3-generalized skew projection. Assume that (a+b)3=-(a+b)*Then

a3+3a2b+3a2b+3ab2+b3=-a*-b*hence a2b+ab2=0,ab(a+b)=0therefore ab =0 then ab=ba=0.

Theorem 1.6. Let a,b$\in$ M(β)be 3-generalized skew projection and a= h+ik(h,k,hk$\in$ H(β)). Then b-a is a 3-generalized skew projection if and only if a2b=ab2.

Proof. Fist assume that ab=ba, a2b=ab2then (b-a)3=b3-3a2b+3ab2-a3=

b3-a3=-b*+a*= -(b-a)*$\Rightarrow$ (b-a)3=-(b-a)*.Hence b-a is a 3- generalized skew projection. Now assume that (b-a)3=-(b-a)*.It follows that b3-3a2b+3ab2-a3-3a2b+3ab2-a3=-b*+a*then 3a2b=3ab2hencea2b=ab2.

Theorem 1.7. Let that a= h+ik$\in$ M(β) is a 2-generalized skew projection. Then

(i) a is normal.

(ii) $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \lambda^3 = -1\} \cup \{0\}$.

(iii) if hk $\in$ H (β), then

(a*)2-a and a4=-a.

Proof. (1) since a2=-a* hence aa*=a(-a2)=(-a2)=(-a2)a=a*a. (2) Let $\lambda \in \sigma(a)$, then $\lambda = \chi(a)$ for some $\chi \in \Delta_a$, hence b proposition 1.3 $\bar{\lambda} = \chi(a^*) =$ $\chi(-a^2) = -\chi 9a)^2 = -\lambda^2$.If $\lambda \neq$, then $|\lambda|$ =1and $\lambda^4 = \lambda\lambda^3 = \lambda(-\bar{\lambda} = -\lambda\bar{\lambda} - 1$.

(3) From a2=-a*then we have;h2+h-k2=i(k-2hk).Proposition 1.3 (4) and propposituin 1.1 (2) show that $\sigma(h^2+h-k^2), \sigma(k-2hk) \subseteq R$ hence $\sigma(k-2hk) \subseteq R \cap iR = \{0\}$.Thus $r(k-2hk) = 0$. since $hk \in H(A), k-2hk \in H(A)$. Now use proposition 1.1(2) to get k= 2hk, hence h2-k2=-h.Therefore (a*)2=(h-ik)2=h2-2ihk-k2=-h-2ihk= -h-ik=-a and a4=(a2)2=(-a*)2=-a.

Theorem 1.8. Suppose a=h+ik$\in M(\beta)$ and suppose that

$$\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \lambda^3 = -1\} \cup \{0\}$$

We have

(i) If a is normal, then r(a2+a*)=0.

(ii) If hk, h2,k2$\in$ H(β) then a is a 2-generalized skew projection.

DOI: 10.22034/cmes.2023.2018673.1019

Proof. Let b= a2+a* (1)Since a is normal, hk=kh, thus b $\in$ B(a).Take $\lambda \in \sigma(b)$. Then λ = $\chi(b)=\chi(a)^2+\chi(a^*)$ for some $\chi \in \Delta_a$, Casel: $\chi(a)=0$. Then we have $\lambda=0$. Case 2: $\chi(a)\neq 0$. Sine $\chi(a)\in\sigma(a),\chi(a)^3=-1$ it follows that $\lambda\chi(a)=\chi(a)^3+\chi(\bar{a})\chi(a)=-1+1=0$ and so $\lambda=0$. Hence $\sigma(b)=\{0\}$.

(2) Since h,k,h2, hk and k2are all hermitian, it followfrom [1,theorem 2.14] that hk=kh. Thus a is normal. From b= a2+a*=h2+2ihk-k2+ (h-ik)

We get i (2hk-k) =b-(h2+h-k2). By proposition 1.1 and proposition 1.3 $\sigma(2hk-k)\subseteq R\cap iR=0$. Since 2hk-k $\in H(A), 2hk-k=0, b=h^2+h-k^2 \in$ H(A). Since by (1), r(b)=0, we conclude that b=0,hence –a2=a*.

Corollary 1.9 For a = h+ik $\in M(\beta)$ with h,k,hk,h2,k2 $\in H(\beta)$ the following assertions are equivalent;

(1) a is a normal partial isometry and a4=-a.

(2) a is normal and a4= -a.

(3) $\sigma(a)\subseteq\lambda\in\{\square : \lambda^3=-1\}\cup\{0\}$.

(4) a2=-a*.

Proof. •(1) $\Rightarrow (2)$: clear.

• (2) $\Rightarrow$ (3):Use the spectral mapping theorem [7,theorem 10.28]

• (3) $\Rightarrow$ (4): theorem 2.2. (2).

• (4)$\Rightarrow$ (1): By theorem 2.1 (1), a is normal, and part (3) of theorem 2.1 give aa*aa(-a2)a=-a4=a.

Theorem 1.10. Let a,b $\in M(\beta),(a+b\neq 0)$ be 2-generalized skew projection then a+b is a 2- generalizded skew projection if and only if ab=ba0.

Proof. We have (a+b)2=a2+ ab+ ba+b2=-a*+ab+ba-b*.If ab= ba=0

then (a+b)2=-a*-b*=-(a*+b*)=-(a+b)*. Therefore a+b is a 2-generalized skew projection. Conversely, Assume that (a+b)2=-(a+b)*.

Then a2+ab+ba+b2=-a*-b*=a2+b2.Then ab+ba=0.Applying a on the left and on the right gives a2b+aba=0=aba+ba2. Then –a*b+aba= 0= aba-ba* hence –a*b=-ba*. Since a is normal, a* is normal. Proposition 2.1 in [8] implies now that ab=ba,thus ab= ba=0.

Theorem 1.11. Let a,b $\in M(w)$ and be 2-generalized skew projection a= h+ik(h,k $\in H(W), hk\in H then (b-a)^*=-(b-a)^2 \Leftrightarrow ab=ba=-a^*$.

Proof. Fist assume that ab=ba=-a*then (b-a)2=b2-ba-ab+a2=b2+a*+a*+

a2=b2+2a*-a*=b2+a*=-b*+a*=-(b*-a*)then (b-a)2=-(b*-a*). It follows that b2-ba+a2=-b*+a*hence 2a2=2ba hence a2= ba then-a*=ba= ab.

Theorem 1.12 Let a.b $\in M(W)$, where a is normal. Then a*a=a*b,aa*=ba* if and only if a2=ab=ba.

Proof. Suppose a*a=a*b,aa*= ba* and a is normal. Hence a*a=a*b=ba*.

By Putnam Fugledes theorem [4] ab= ba and a2= ab then a2=ab=ba.

Suppose b $\in$ M(W) is 2-generalized projection. Then b*b=b2b=bb2=

bb*. So b is normal. Also (b*b)2=(b*)2b2=b4b2=bb*=b*b.Hence b is a partial isometry and b*b is an orthogonal projection onto R(b)=R(b*), where R(b) denotes the rang of an operator b.

Theorem 1.13 Let a,b $\in$ M(w) be two generalized projection.Then b-a is a generalixed projection if and only if aa*=a*b.aa*=ba*.

Proof. Suppose b-a is a generalized projection.. Then (b-a)2= (b-a)*.

Hence b2- ab-ba+a2=b*-a*.So 2a*=ab+ba. Hence 2a2=ab+ba.Thus 2a3= aba+ba2=a2b+aba.So ba2=a2b.Since a is generalized projection, ba*=a*b.By Fugledes theorem, [7], ab= ba. Hence a2=ab=ba. Hence by theorem 2.1 a*a=a*b,aa*=ba*.Suppose a*a=a*b,aa*=ba*.Then by theorem 2.1 a*=a2=ab=ba. Hence 2a*=ab+ba.Then by premultiply and postmultiply the last equation by a*we get aba*+ baa*=a*ab+a*ba.

While by premultiply and postmultiply the last equation by a* we get ab=

a*ba*+abaa*-aa*ba,ba=a*aba+a*ba*-aba*a respectively. Since aa*=a*

a,ab+ba=2a*ba*=2a*.Hence b-a is a generalized projection.

Acknowledgements: The author express his gratitude to the referee for carefully reading of the manuscript and giving useful comments.

REFERENCES

[1] F.F.Bonsall and J.Duncan, Nurerical Ranges of Operators on Normed Spaces and Elements of Normed Algebras, Cambridge University Press, (1971).

[2] H.Du and Y.Li, The spectral characterization of generalized projections, Linear Algebra Appl.400,(2005), 313-318.

[3] L.Lebtahi and N. Thome, A note on R-generalized projection, Linear Algebra, Appl. 420. (2007), 572-575.

n-GENERALIZED SKEW PROJECTIONS

1 DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, ISLAMIC AZAD UNIVERSITY CENTRAL TEHRAN BRANCH, TEHRAN, IRAN.

E-mail address:msasgari @yahoo.com moh.asgari @ iauctb.ac.ir

2DEPARTMENT OF MATHEMATICS. FACULTY OF SCIENCE, ISLAMIC AZAD UNIVERSITY E-mail address: s_karimizad@ yahoo. Com.